

Remark on the strength of singularities with a C^0 metric

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Abstract

Recently Nolan constructed a spherically-symmetric spacetime admitting a spacelike curvature singularity with a regular C^0 metric. We show here that this singularity is in fact weak.

In a recent paper Nolan [1] constructed a simple spherically-symmetric spacetime which includes a spacelike curvature singularity with a continuous (C^0) metric. The goal was to use this example to demonstrate that a curvature singularity with a C^0 metric may be strong (according to the classification by Tipler [2] and by Ellis and Schmidt [3]). In this note we shall show that this singularity is in fact *weak*. We prove this by solving the second-order differential equation for the norm a of the radial Jacobi field, Eq. (n2) (hereafter the letter n before the equation number refers to Nolan's paper [1]).

We shall use here the notation of Ref. [1]. It will be assumed that the dynamics of $a(t)$ and $x(t)$ is correctly described by the corresponding second-order differential equations, i.e. Eq. (n2) for $a(t)$ and the equation preceding Eq. (n6) for $x(t)$.

Since $f = f(x)$ (with $x = u + v$), in Eq. (n2) we substitute $f_{uv} = f''$. We first show that

$$a(t) = \dot{x}e^{-2f} \equiv \bar{a}(t) \tag{1}$$

is an exact solution of Eq. (n2). To demonstrate this, we differentiate Eq. (1):

$$\dot{a} = (\ddot{x} - 2\dot{f}\dot{x})e^{-2f} = (\ddot{x} - 2f'\dot{x}^2)e^{-2f} , \quad (2)$$

where we have used $\dot{f} = f'\dot{x}$. From the differential equation for $x(t)$ [the one preceding Eq. (n6)] we then find

$$\dot{a} = -2f' = -2\dot{f}/\dot{x} , \quad (3)$$

and hence $\ddot{a} = -2(\dot{f}/\dot{x})$. Now, since

$$f_{uv} = f'' = \dot{x}^{-1}(\dot{f}/\dot{x}) , \quad (4)$$

one can easily verify that Eq. (n2) is satisfied.

We now use the Wronskian method to construct a second independent solution. Since the Wronskian of Eq. (n2) is a constant, this second solution takes the form

$$a(t) = \bar{a}(t) \int_t^t \bar{a}(t')^{-2} dt' \equiv \hat{a}(t) . \quad (5)$$

Consider now the radial Jacobi field which vanishes at $t = t_1$. Its norm a is a linear combination of $\bar{a}(t)$ and $\hat{a}(t)$ which vanishes at t_1 , so it must take the form

$$a(t) = A \bar{a}(t) \int_{t_1}^t \bar{a}(t')^{-2} dt' \equiv \tilde{a}(t) . \quad (6)$$

Here A is a non-vanishing constant, and without loss of generality we may take $A = 1$.

Both \dot{x} and e^{-2f} are finite and strictly positive in the interval $t_1 \leq t \leq 0$, and so is $\bar{a}(t)$. Consequently, $\tilde{a}(t)$ is finite and non-vanishing everywhere at $t_1 < t$, and particularly at $t = 0$. Thus, the singularity at $t = 0$ is weak.

It seems that the error in Ref. [1] results from a misuse of the WKB method in the present case. Namely, the inequality before Eq. (n3) does *not* imply that a will either vanish or diverge at $t = 0$. To illustrate this by a simple example, consider the equation $\ddot{a} + F(t)a = 0$ with $a(t) = 1 + t \sin(1/t)$. Then near $t = 0$, $a \cong 1$ and

$$F(t) = -\ddot{a}/a \cong -\ddot{a} \cong t^{-3} \sin(1/t) , \quad (7)$$

so the inequality before Eq. (n3) is satisfied (to the same extent that it is satisfied in Ref. [1]; i.e. the limit does not exist), and yet $a(t)$ is continuous and nonvanishing at $t = 0$.

It should be pointed out that Nolan is basically correct in his claim that Tipler's definition of weakness is not precisely equivalent to the existence of a non-singular C^0 metric. (The association of weakness with a C^0 metric in Ref. [4] resulted from a misinterpretation of a statement in Ref. [2].) It is not difficult to construct examples of a singular hypersurface with a non-singular C^0 -metric, such that the singularity is not entirely weak. Such singularities have a more complex structure, however, and typically the singularity is strong on subsets of zero measure only (i.e. on points, lines, or two-surfaces). The present author is not aware of any example of a singular hypersurface with a non-singular C^0 metric, such that the singularity is strong in the entire hypersurface (or even in an open subset of it).

The strength of the null curvature singularity inside a spherical charged black hole [5], inside a spinning black hole [4], and in the class of solutions constructed by Ori and Flanagan [6], was analyzed independently of the continuity of the metric tensor. This analysis was based on the divergence rate of curvature as a function of *proper time* (as was mentioned explicitly in Ref. [5]). In all these cases, the singularity was found to be weak (according to Tipler's definition). The details of this analysis will be presented in a separate paper.

References

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